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Quantum gates and degenerations

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Abstract

A three state Hamiltonian with a degenerate ground state is shown to exhibit the property of generating quantum gates for fixed values of the energy parameters just by using the ability of this quantum system to evolve in time yielding suitable sets of quantum probabilities. A method is presented, based on the conventional rules of quantum mechanics, for obtaining several gates without changing the system configuration, i.e. just by adjusting the times in which the interaction is switched on and off. The mathematical rules for generating such a set of gates are completely defined and described in this work. The corresponding physical implementation should involve fast Rabi pulses and very precise switches of the kind proposed in quantum geometric computation.

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1. Introduction

Quantum computation has become an extremely active field of research in the last decade. Important avenues of this entirely new field of physics are: (a) the possibility of implementing efficient algorithms which are more powerful than those so far known in classical computation, (b) the new properties for computation that emerge from the roots of quantum mechanics, (c) a new theory of information based on the concept of the qubit, and (d) a physical basis for the theory of computation. All these fields of activity point towards the development of a new and strong tool for science [1–3].

Despite all these developments, some basic aspects remain unresolved. The central point is undoubtedly the feasibility of physical implementation. Many different ideas have been put forward without obtaining substantial progress so far. It is widely known that the problem of understanding and defining *decoherence* is central to physical implementation. Quantum computers, if they can ever be constructed, would require a physical system to play the *qubit hardware* role and then to be transformed under the action of the quantum logic gates in order to bring the *quantum software* to work.

The transformations on qubits are then the crucial point to study. These transformations are very often represented by Hermitian operators governed by the laws of quantum mechanics.

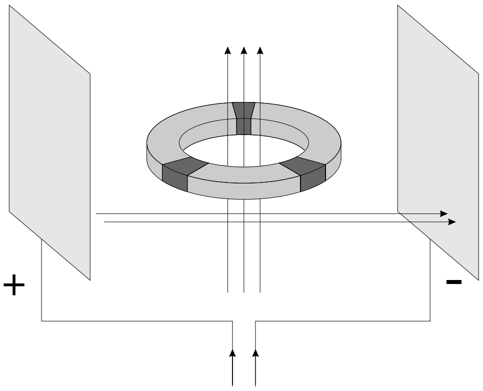

Figure 1. Aharonov–Bohm persistent-current rings scheme.

These operators make the quantum gates. Therefore, we look for operators able to implement the highest possible number of quantum gates with the minimal modifications in the parameters they can depend upon just in order to maximize the avoidance of errors.

Can a maximal number of quantum logic gates be constructed just by using the simplest possible operators? This is the main goal of this paper for single-qubit gates. A three state Hamiltonian is studied trying to explore how many quantum gates can be implemented by using the simplest rules of quantum mechanics.

The proposed system is a three-dimensional Hilbert space with two energy levels: a doubly degenerate ground state and an excited state. Since a qubit is a two-level quantum system its role is played by the degenerate states. The other one is an auxiliary state. Furthermore an interaction that breaks up the degeneracy is introduced and allows amplitude interchanges between all the three states. Experimentally the degenerate three state idea can be realized using Aharonov–Bohm persistent-current rings. Persistent-current states in normal (not necessarily superconducting) metals have been found in [4, 5]. They appear as a physical outcome of the Aharonov–Bohm effect produced by a magnetic flux threading a loop formed by a three island ring in which the electron is allowed to hop from one site to the other. After stabilization of the persistent-current configuration, the device is affected by a static electric field applied perpendicular to the magnetic field. Quantum computation schemes can thus be realized either by using the Aharonov–Bohm persistent-current states in non-superconducting structures of small size [6] or alternatively in superconducting systems (using the Josephson effect) operating with macroscopic quantum interference effects such as those appearing in SQUIDS [7]. The first option is illustrated in figures 1 and 2. It is worth noting that this scheme can be used in any physical system where in order to avoid undesirable interactions the three levels are located sufficiently far from the rest of the spectra.

The rest of this paper is structured as follows: in section 2 the evolution of a general state under this Hamiltonian is obtained. This allows us to establish the condition for a quantum gate transformation. Then it is shown how to obtain different quantum gates in the simplest way: changing the potential strength. With this result it is possible to show how the system works.

In section 3 the main result is presented: the construction of quantum gate sets without modifying the parameters of the Hamiltonian. The main idea is to use the quantum evolution of the system as the generator of the quantum gates such that at *different times* we have *different gates*. It is shown that it is just necessary *to wait* in order to have different quantum gates instead of changing the interaction parameters.

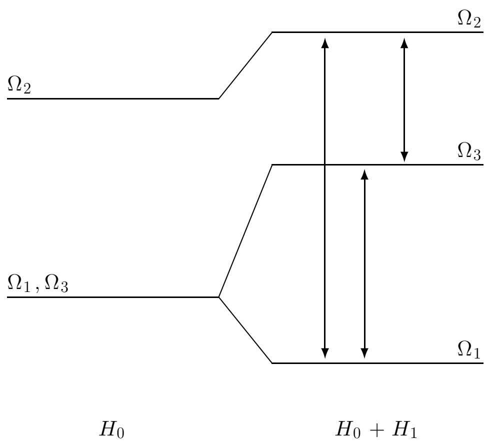

Figure 2. Energy level representation. Interaction and probability decays between the three states.

Section 4 is a generalization of sections 2 and 3. One can indeed summarize all possibilities of the system in a systematic way. Finally it is established how the introduction of a new non-commutative interaction allows a very accurate approximation for any single quantum gate.

2. The quantum gate condition

The evolution of a general state in our three state system is studied. The free Hamiltonian in the energy states basis is diagonal, while the interaction Hamiltonian must obviously be of the off-diagonal type. Let us write them as follows:

$$\mathbf{H_0} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{1}$$

$$\mathbf{H_1} = a \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}. \tag{2}$$

By means of the interaction, the excited state transfers amplitude from one state of the qubit to the other. The behaviour of the system is easily found by solving the Schrödinger equation. Using the free energy states basis, a general state can be written as

$$|\psi(t)\rangle = C_1(t)|\epsilon_1\rangle + C_2(t)|\epsilon_2\rangle + C_3(t)|\epsilon_3\rangle. \tag{3}$$

The Schrödinger equation gives

$$\mathrm{i}\hbar\partial_t|\psi(t)\rangle = (\mathbf{H_0} + \mathbf{H_1})|\psi(t)\rangle \tag{4}$$

which leads to a system of three differential equations,

$$\begin{aligned} \mathrm{i}\dot{C}_1(t) &= -C_1(t) + aC_2(t) + aC_3(t) \\ \mathrm{i}\dot{C}_2(t) &= +aC_1(t) + 2C_2(t) + aC_3(t) \\ \mathrm{i}\dot{C}_3(t) &= +aC_1(t) + aC_2(t) - C_3(t). \end{aligned} \tag{5}$$

The general solutions of the system are linear combinations of exponentials of the form:

$$C_i(t) = \sum_i A_i \exp\{\mathrm{i}\Omega_i t\} \tag{6}$$

where Ω_i are the eigenvalues of the Hamiltonian. These are obtained as usual:

$$M = \begin{bmatrix} -\Omega - 1 & a & a \\ a & -\Omega + 2 & a \\ a & a & -\Omega - 1 \end{bmatrix} \qquad \mathrm{Det}\, M = 0 \tag{7}$$

giving a degenerate cubic equation on Ω:

$$\Omega^3 - 3(1 + a^2)\Omega - 2(1 + a^3) = 0. \tag{8}$$

The solutions are three different eigenvalues depending on a. They clearly show that the interaction breaks up the degeneracy

$$\Omega_1 = \tfrac{1}{2}(1 + a) - \tfrac{3}{2}\sqrt{1 + a^2 - \tfrac{2}{3}a} \tag{9}$$

$$\Omega_2 = \tfrac{1}{2}(1 + a) + \tfrac{3}{2}\sqrt{1 + a^2 - \tfrac{2}{3}a} \tag{10}$$

$$\Omega_3 = -(1 + a). \tag{11}$$

The definition of the following variables simplifies the calculations:

$$\omega_0 = \tfrac{1}{2}(1 + a) \tag{12}$$

$$\omega_1 = \tfrac{3}{2}\sqrt{1 + a^2 - \tfrac{2}{3}a}. \tag{13}$$

The eigenvalues appear as

$$\Omega_1 = \omega_0 - \omega_1 \tag{14}$$

$$\Omega_2 = \omega_0 + \omega_1 \tag{15}$$

$$\Omega_3 = -2\omega_0. \tag{16}$$

In order to obtain the solutions of equations (5) it is necessary to fix the initial conditions of the physical system. As it has been already explained the two degenerate states in the absence of interaction form the qubit, i.e. the qubit is the subsystem $\{|\epsilon_1\rangle, |\epsilon_3\rangle\}$. Also a quantum gate is a transformation on a qubit. For this reason the initial state of the system must represent an initial state of the qubit which is obtained with an initial zero amplitude for the excited state.

Then we look for solutions of the system of equations (5) with initial conditions of the form:

$$|\psi(a, 0)\rangle = \{C_1(0), 0, C_3(0)\} \tag{17}$$

where the pair $\{C_1(0), C_3(0)\}$ is the desired configuration for the initial state of the qubit[3]. Taking this into account and performing a few calculations, the probabilities of finding each

[3] For simplicity in the calculations the initial conditions are written as

$$C_1(0) = c_1\, \mathrm{e}^{\mathrm{i}\mu_1}$$
$$C_3(0) = c_3\, \mathrm{e}^{\mathrm{i}\mu_3}$$
$$\tan\delta = \frac{2c_1c_3\sin(\mu_1 - \mu_3)}{2c_1^2 - 1}$$

where c_j, μ_j are real parameters, and the normalization condition $c_1^2 + c_3^2 = 1$ holds.

state at a given time are given by

$$|C_1(t)|^2 = \frac{1}{2} - \frac{1}{2}(1+2c_1c_3)\left(\frac{a}{\omega_1}\right)^2 \sin^2(\omega_1 t) + \frac{\left(2c_1^2-1\right)}{2}\left[\cos(\omega_1 t+\delta)\cos(3\omega_0 t) + \frac{(3-a)}{2\omega_1}\sin(\omega_1 t+\delta)\sin(3\omega_0 t)\right] \quad (18)$$

$$|C_2(t)|^2 = (1+2c_1c_3)\left(\frac{a}{\omega_1}\right)^2 \sin^2(\omega_1 t) \quad (19)$$

$$|C_3(t)|^2 = \frac{1}{2} - \frac{1}{2}(1+2c_1c_3)\left(\frac{a}{\omega_1}\right)^2 \sin^2(\omega_1 t) - \frac{\left(2c_1^2-1\right)}{2}\left[\cos(\omega_1 t+\delta)\cos(3\omega_0 t) + \frac{(3-a)}{2\omega_1}\sin(\omega_1 t+\delta)\sin(3\omega_0 t)\right]. \quad (20)$$

The probabilities verify the normalization condition for all times:

$$|C_1(t)|^2 + |C_2(t)|^2 + |C_3(t)|^2 = 1. \quad (21)$$

Our purpose is to find quantum logic gates. Therefore we need the conditions under which the Hamiltonian behaves as one of the characteristic logic gates. Following the previous argument the final state we are interested in must be of the same kind: zero amplitude for the excited state. As the interaction mixes the amplitudes between all the states this is the condition that tells us when the action of a quantum gate takes place. The necessary information is recorded in equations (18)–(20). Applying the previous condition the following typical times for the quantum gates are obtained:

$$C_2(V,t) = 0 \quad \Rightarrow \quad t_n = \frac{\pi}{\omega_1} n \quad (22)$$

with n being integer. It is worth noting that this condition holds independently of the initial qubit configuration which means that a quantum gate is properly defined by the pair: value of the potential and time: $\{a, t_n\}$.

Substituting the time (22) in (18)–(20) we obtain a very simple probability expression for the instants where a quantum gate *acts*[4]. The dependence now lies on the integer n:

$$|C_1(n)|^2 = \frac{1}{2} + \frac{\left(2c_1^2-1\right)}{2}\left[\cos\left(\frac{3\omega_0}{\omega_1}\pi n\right)\cos(\pi n)\right] \quad (23)$$

$$|C_2(n)|^2 = 0 \quad (24)$$

$$|C_3(n)|^2 = \frac{1}{2} - \frac{\left(2c_1^2-1\right)}{2}\left[\cos\left(\frac{3\omega_0}{\omega_1}\pi n\right)\cos(\pi n)\right]. \quad (25)$$

With these equations we are able to investigate the quantum logic gates that we can construct with our Hamiltonian. Changes in the strength of the interaction lead to different populations of the qubit and hence to different quantum gates. The first question that arises is how many different quantum gates one can construct within this framework by means of a potential of relative strength a.

There is another and perhaps more powerful source of quantum gates: time. As it is easy to see from (23) and (25) the populations depend on n (t_n). This implies that for the same

[4] We set $\delta = 0$ because it is usual in quantum computation to use a pure base state as playing the role of the initial state $|\epsilon_1\rangle \equiv |0\rangle \to (1,0)$ or $|\epsilon_3\rangle \equiv |1\rangle \to (0,1)$.

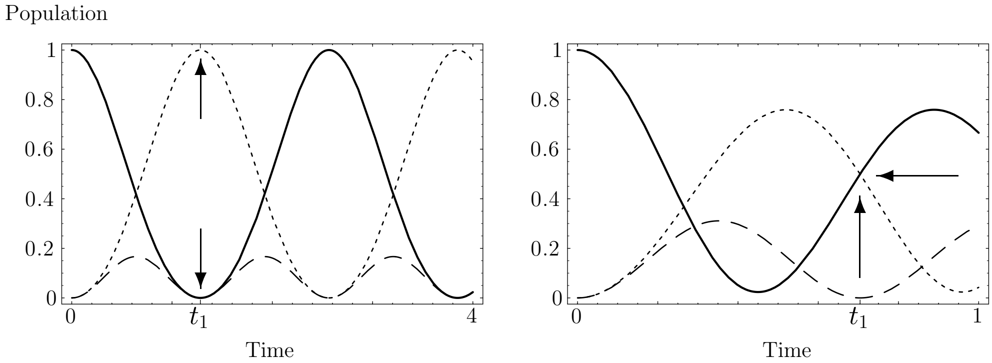

Figure 3. Plot of the populations for each state as a function of time for $a = -1$ (bit-flip gate) (*a*) and $a = -\frac{1}{9}(2\sqrt{22} + 13)$ (Hadamard gate) (*b*). The solid line represents the probability for state 1, the dashed line for state 2 and the dotted line for state 3.

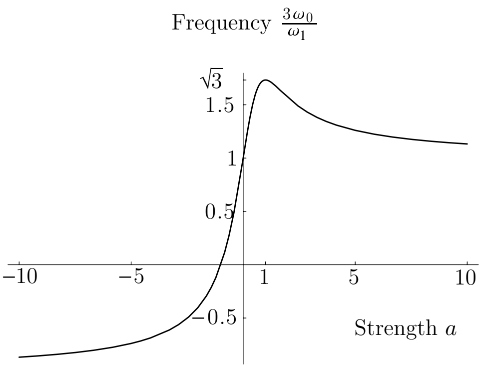

Figure 4. Dependence of the frequency $\frac{3\omega_0}{\omega_1}$ with the strength of the interaction a. We see that there exists at least one value of a that yields a frequency between 0 and 1.

value of a we can have different quantum gates if we let the system evolve a time t_{n_1} or t_{n_2}. The question that now arises concerns the number and properties of the quantum gates that can be constructed for a fixed potential just by choosing different times. Let us see how this works with some examples. Firstly we fix our attention on changes of the potential. The simplest cases have already been presented in [8]: the bit-flip gate (figure 3(*a*)) for $a = -1, t = \frac{\pi}{\sqrt{6}}$, and the Hadamard gate (figure 3(*b*)) for $a = -\frac{1}{9}(2\sqrt{22} + 13), t = \frac{3\pi}{2\sqrt{26+4\sqrt{22}}}$.

In order to answer the first question on the accessible quantum gates by varying the strength of the interaction we look at equations (23) and (25) and see that we are able to get any desired value for the final population of the basis states if the frequency $\frac{3\omega_0}{\omega_1}$ varies between 0 and 1. This is always possible as is shown in figure 4. Therefore with a judicious choice for a we can construct any quantum gate yielding any final configuration for the probabilities.

Note that we are referring always to the probabilities instead of the amplitudes. The reason is that we are not able to get any desired value for the final amplitude and then the Hamiltonian does not allow the construction of any arbitrary quantum gate. For example, the phase-shift gate cannot be obtained with this Hamiltonian.

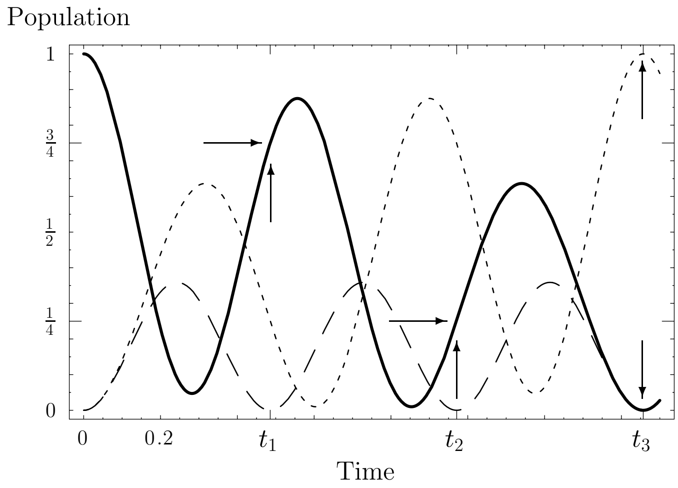

Figure 5. Plot of the populations for each state as a function of time for $a = -\frac{1}{15}(4\sqrt{46} + 31)$. Arrows show the quantum gate transformation. The solid line represents the probability for state 1, the dashed line for state 2 and the dotted line for state 3.

Let us now look for another gate. For example we set the initial qubit state to $|\epsilon_1\rangle \equiv |0\rangle$ and we want a quantum gate that gives a final probability of $\frac{3}{4}$ for the initial base state and consequently $\frac{1}{4}$ for the other. From equation (25) the frequency $\frac{3\omega_0}{\omega_1}$ may be $-\frac{2}{3}$. This implies $a = -\frac{1}{15}(4\sqrt{46} + 31)$. The result is shown in figure 5. With these examples it is clear how to construct quantum gates by modifying the potential strength. We now turn our attention to another interesting property of our Hamiltonian using a final example.

3. Multiple quantum gates with fixed Hamiltonian

The previous figure (figure 5) shows how *time* becomes a source of quantum gates. Looking at the zeroes of the second state populations we observe different populations for the qubit states: for $n = 1 \to \left(\frac{3}{4}, \frac{1}{4}\right)$, for $n = 2 \to \left(\frac{1}{4}, \frac{3}{4}\right)$ and for $n = 3 \to (0, 1)$. This means that with the same Hamiltonian we can produce three different quantum gates. In order to use them appropriately what we have to do is just to apply the interaction at a specific time, depending on the quantum gate we need.

Now we can ask about the second question of how many quantum gates we can have and which are those for a fixed value of the parameters of the Hamiltonian. As we have already seen we cannot obtain all single quantum gates (i.e. the phase-shift gate, for example). But we will show that in one Hamiltonian we can condensate as many quantum gates as we want. It is worth distinguishing two different cases for values of $3\frac{\omega_0}{\omega_1}$:

- Rational frequency rate.
- Irrational frequency rate.

Each situation gives rise to disjoint sets of quantum logic gates. A quantum gate obtained with rational frequency is not accessible with an irrational one although we will see that we can approximate any single quantum gate by means of irrational frequencies. As this is the most interesting property of the irrational frequencies we postpone the discussion until the following section. We shall be dealing just with the rational case for the moment.

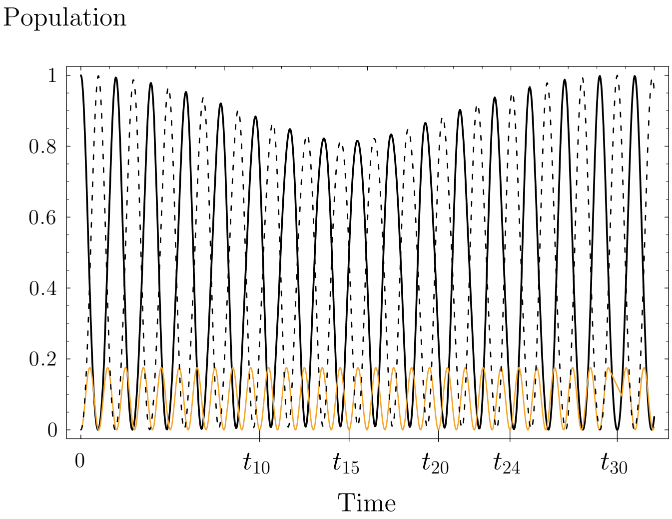

Figure 6. Plot of the populations for each state as a function of time for the frequency $\frac{3\omega_0}{\omega_1} = -\frac{1}{30}$ and $a \simeq -1.055\,97$. The zeroes of the second state population fix the times for the appearance of the quantum gates. In this case we have 30 different gates. The solid line represents the probability for state 1, the dashed line for state 2 and the dotted line for state 3.

The ratio of frequencies $\frac{3\omega_0}{\omega_1} \in \mathcal{Q}$. We can write the frequency as $\frac{m}{l}$ with $m, l \in \mathcal{N}$ irreducible. Then:

- Looking at equations (23) and (25), we see that because of the periodicity of the cosine the set of quantum gates that we obtain for a fixed ratio of frequencies is finite. The maximum number of different gates will be l as some of the gates in the interval $n \in \{1, l\}$ can be repeated. For arbitrary n the number of generated quantum gates is equal to n mod l.
- We could have in one Hamiltonian all the desired gates of this kind. What we have to do is to look for the frequencies $\left(\frac{3\omega_0}{\omega_1}\right)$ yielding each desired gate. Then we multiply their denominators to form the final frequency[5]. Next we obtain the corresponding value for the strength of the interaction from (12) and (13). With this value of a we have all the quantum gates in the fixed Hamiltonian.

  For example, we want a transformation that takes the initial configuration $(1, 0)$ to $\left(\frac{1}{8}(3 - \sqrt{5}), \frac{1}{8}(5 - \sqrt{5})\right)$. This is obtained with a ratio of frequencies equal to $-\frac{1}{5}$. Now we would like to construct with the same Hamiltonian the transformations that we have already presented: bit-flip, Hadamard, $\frac{3}{4}$-gate, with frequencies $0, -\frac{1}{2}, -\frac{2}{3}$, respectively. We multiply together all the denominators of these frequencies. In order to have the bit-flip gate we make sure that the result is even.

The resulting ratio of frequencies is $-\frac{1}{30}$. The corresponding strength for the interaction is $a = -\frac{2701 + 2\sqrt{5398}}{2697} \simeq -1.055\,97$ (figure 6). With these conditions we have a Hamiltonian that gives us a set of 30 different quantum gates, among them those in which we are interested. The times at which we have quantum gates are given by $t_n = \frac{851 n\pi}{30\sqrt{5402 + 4\sqrt{5398}}}$. The times

[5] We can modify this frequency multiplying it by a number coprime to the obtained denominator keeping the gates we wanted to operate. This possibility gives us a very useful tool for time control. Depending on the needed time orders we calculate the appropriate order for the strength a from equation (22) for t_n. Then we adjust the frequency to be accessible with the new strength order. This is also a good tool for the application of this scheme to three level systems of different nature.

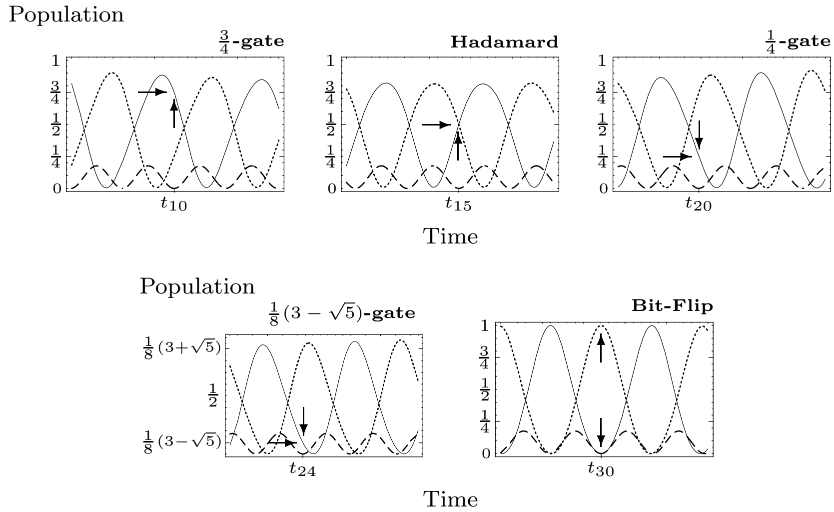

Figure 7. Magnification of the results of figure 6 the populations for each state as a function of time for the frequency $\frac{3\omega_0}{\omega_1} = -\frac{1}{30}$ and $a \simeq -1.055\,97$. The zeroes of the second state population fix the times for the appearance of the quantum gates. The solid line represents the probability for state 1, the dashed line for state 2 and the dotted line for state 3.

$t_1 \simeq 1.1808$ between two adjacent gates are of the order of 10^{-12} s ($\hbar/\eta$ where η is the free Hamiltonian energy, around 1 meV). This is fast enough for the transformation to take place (for $n = 30$ the change in time is small). Also it is worth noting that despite having a lot of gates in one Hamiltonian in each operation we apply just one transformation. Then we switch off the interaction and this allows us to increase the time for the transformation because the same transformations are repeated cyclically. Another time modification can be made through the modification of frequency as we have pointed out before. We have the following gates depending on a specific time (figure 7):

$n = 10$	$\frac{3}{4}$ − gate	$n = 24$	$\frac{1}{8}(3 - \sqrt{5})$ − gate
$n = 15$	Hadamard	$n = 30$	bit-flip
$n = 20$	$\frac{1}{4}$ − gate.		

With this mechanism we have a great flexibility to apply distinct quantum gates to a qubit. Without varying the interaction that acts over the qubit (which can always be a potential source of error) we can use the applied unitary transformation by controlling the time.

With the help of a more complicated frame and with a complete set of quantum gates this idea also opens the door to a programmable quantum computer. This is because programming an algorithm means to program the time of application of the interaction in each site. So without changes in the physical structure we can make different calculations.

4. Evolution operator: rotation matrix

We have already shown with simple examples the idea of having multiple quantum gates in a unique Hamiltonian. We now focus our attention on the evolution operator in order to exploit this property and to find the maximum number of quantum gates that we can have.

The evolution operator turns out to be quite simple with many symmetric elements which make its treatment easier. We show again the condition that leads to quantum gates for the qubit system. Then for those times we reduce the 3×3 unitary matrix to a 2×2 matrix that provides us with the matrix form of the quantum gates as a function of the integer n. Some simple modifications of this matrix will allow us to show that this matrix corresponds to a rotation around the x-axes of the qubit Hilbert space: the angle depending upon the interaction strength.

As the Hamiltonian does not depend on time the calculation of the corresponding evolution operator is not difficult but tedious. We can proceed either by calculating the exponential $\exp(-\mathrm{i}Ht)$ or by using the eigenvectors to diagonalize the Hamiltonian through a change of basis. After that we perform the exponentiation and undo the basis change obtaining the unitary matrix in the qubit state representation. The result can be written in the form:

$$\mathbf{U}(a,t) = \begin{pmatrix} \alpha(a,t)+\beta(a,t) & \gamma(a,t) & -\alpha(a,t)+\beta(a,t) \\ \gamma(a,t) & \Gamma(a,t) & \gamma(a,t) \\ -\alpha(a,t)+\beta(a,t) & \gamma(a,t) & \alpha(a,t)+\beta(a,t) \end{pmatrix} \tag{26}$$

where each of the matrix elements are linear combinations of exponentials of the eigenvalues. The time dependence is just in the exponentials. We can write explicitly the expressions for the parameters[6]:

$$\alpha(a,t) = \frac{1}{2}\,\mathrm{e}^{\mathrm{i}2\omega_0 t} \tag{28}$$

$$\beta(a,t) = \frac{1}{2+f^2}\left(\mathrm{e}^{\mathrm{i}\omega_1 t} + \frac{f^2}{2}\,\mathrm{e}^{-\mathrm{i}\omega_1 t}\right)\mathrm{e}^{-\mathrm{i}\omega_0 t} \tag{29}$$

$$\gamma(a,t) = \frac{f}{2+f^2}(\mathrm{e}^{\mathrm{i}\omega_1 t} - \mathrm{e}^{-\mathrm{i}\omega_1 t})\,\mathrm{e}^{-\mathrm{i}\omega_0 t} \tag{30}$$

$$\Gamma(a,t) = \frac{f^2}{2+f^2}\left(\mathrm{e}^{\mathrm{i}\omega_1 t} + \frac{2}{f^2}\,\mathrm{e}^{-\mathrm{i}\omega_1 t}\right)\mathrm{e}^{-\mathrm{i}\omega_0 t}. \tag{31}$$

As was pointed out above (17) the initial state is of the form $(c_1, 0, c_3)$ and we will have quantum gates in those times where the population of the second state vanish again. This condition is satisfied for the zeros of the $\gamma(a,t)$ evolution matrix element[7]. It is easy to see from (30) that this occurs for times $t = \frac{\pi}{\omega_1}n$ where n is an integer. This is in agreement with (22). For these times the evolution operator has the form:

$$\mathbf{U}(a,t_n) = \begin{pmatrix} \alpha(a,t_n)+\beta(a,t_n) & 0 & -\alpha(a,t_n)+\beta(a,t_n) \\ 0 & \Gamma(a,t_n) & 0 \\ -\alpha(a,t_n)+\beta(a,t_n) & 0 & \alpha(a,t_n)+\beta(a,t_n) \end{pmatrix}. \tag{32}$$

Now we can reduce this matrix to a 2×2 unitary matrix which acts on the qubit space depending on the parameters a and n. The matrix also represents the quantum operation on the qubit: *the quantum logic gates*. From this operator we can explore all the available gates

[6] Where we have a function of the interaction strength a,

$$f(a) = -\frac{1+a-\sqrt{1+a^2-\frac{2}{3}a}}{1-a+\sqrt{1+a^2-\frac{2}{3}a}}. \tag{27}$$

This function is bounded and continuous and it has upper and lower limits vanishing just for $a = 0$.

[7] Note that the $\Gamma(a,t)$ function need not be zero for the initial condition.

with a fixed value of a, i.e. without changing the system configuration. The gates will be given by

$$\mathbf{U}(a,n)=\begin{pmatrix}\alpha(a,n)+\beta(a,n) & -\alpha(a,n)+\beta(a,n)\\ -\alpha(a,n)+\beta(a,n) & \alpha(a,n)+\beta(a,n)\end{pmatrix}. \tag{33}$$

After some manipulations we shall find a known matrix: the rotation matrix around the $\hat{x}$-axis of the Hilbert space of the qubit. The value of t_n (22) in equations (28) and (29) gives the expression for the new matrix elements:

$$\alpha(a,n)=\frac{1}{2}\exp\left\{2\mathrm{i}\pi\frac{\omega_0}{\omega_1}n\right\} \tag{34}$$

$$\begin{aligned}\beta(a,n)&=\frac{1}{2+f^2}\left(\exp\{\mathrm{i}\pi n\}+\frac{f^2}{2}\exp\{-\mathrm{i}\pi n\}\right)\exp\left\{-\mathrm{i}\pi\frac{\omega_0}{\omega_1}n\right\}\\ &=\frac{1}{2+f^2}\left(\exp\{2\mathrm{i}\pi n\}+\frac{f^2}{2}\right)\exp\left\{-\mathrm{i}\pi\left(\frac{\omega_0}{\omega_1}+1\right)n\right\}.\end{aligned} \tag{35}$$

Taking into account that $\frac{1}{2+f^2}\left(1+\frac{f^2}{2}\right)=\frac{1}{2}$, finally we have

$$\begin{aligned}\alpha(a,n)&=\frac{1}{2}\exp\left\{2\mathrm{i}\pi\frac{\omega_0}{\omega_1}n\right\}\\ \beta(a,n)&=\frac{1}{2}\exp\left\{-\mathrm{i}\pi\left(\frac{\omega_0}{\omega_1}+1\right)n\right\}.\end{aligned} \tag{36}$$

We substitute these expressions in the quantum gate operator yielding:

$$\mathbf{U}(a,n)=\exp\left\{\mathrm{i}\frac{\pi n}{2}\left(\frac{\omega_0}{\omega_1}-1\right)\right\}\begin{pmatrix}\cos\left\{\frac{\pi n}{2}\left(3\frac{\omega_0}{\omega_1}+1\right)\right\} & -\mathrm{i}\sin\left\{\frac{\pi n}{2}\left(3\frac{\omega_0}{\omega_1}+1\right)\right\}\\ -\mathrm{i}\sin\left\{\frac{\pi n}{2}\left(3\frac{\omega_0}{\omega_1}+1\right)\right\} & \cos\left\{\frac{\pi n}{2}\left(3\frac{\omega_0}{\omega_1}+1\right)\right\}\end{pmatrix}. \tag{37}$$

It is well known that this matrix represents the rotation operator about the $\hat{x}$ axis of the qubit Hilbert space up to a global phase. The angle of rotation is $\theta=\pi\left(3\frac{\omega_0}{\omega_1}+1\right)n$ so we write this matrix as $R_x(\theta)$ and the global phase. This means that the quantum gates which act over the qubit during the time evolution can be brought together in one expression, namely

$$\mathbf{U}(a,n)=\exp\{\mathrm{i}\gamma\}R_x(\theta) \tag{38}$$

where

$$\gamma=\frac{\pi}{2}\left(\frac{\omega_0}{\omega_1}-1\right)n \tag{39}$$

$$\theta=\pi\left(3\frac{\omega_0}{\omega_1}+1\right)n. \tag{40}$$

It is now easy to see that for a given value of a we can have many different quantum gates while time is running. Remember that each integer n signals a different instant of time. Therefore, if the ratio of frequencies is rational there will be a time for which the rotation matrix returns to the identity after going through all the gates. Thus the finite set of different quantum gates covered by the evolution operator will be completed.

If the ratio of frequencies is not rational, the rotations never falls into the identity and the circuit is not closed. In this case we have infinite quantum gates.

We can conclude that the most remarkable advantage of our system is that it provides multiple quantum gates with the same configuration. It does not constitute at all a complete system of single gates as not all are reachable: the phase gate for example is not achieved. Even with this drawback it is interesting to have the possibility of achieving different quantum gates controlled by the time evolution itself, because the reduction in the manipulation of parameters is consistent with the desired reduction of errors.

5. Universal single quantum gates approximation

In the previous sections we have shown how to generate quantum gate sets without modifying the value of the interaction. We have emphasized the fact that rational rates of frequencies yield finite quantum gate sets which are cyclically repeated. Thus we have shown how the evolution of the qubit subsystem is governed by a rotation matrix in accordance with the method of quantum geometric computation [9]. Now we would like to turn our attention to the problem of generating an arbitrary rotation around the $\hat{x}$-axis in this context in order to make the association *quantum gates* $\rightarrow$ $\hat{x}$*-rotation* technically possible. The answer is that we cannot generate in general an arbitrary rotation of this kind, as the sets from rational and irrational frequencies are disjoint. Nevertheless in the case of irrational frequency rates we can approximate any such rotation with the desired accuracy because any rotation around an axis of angle θ, being an irrational multiple of 2π, can be approximated with arbitrary accuracy to a previously chosen rotation. This means that selecting a such that $\frac{3\omega_0}{\omega_1}$ is an irrational multiple of 2π, we get a fixed Hamiltonian able to act approximately as any given rotation. From this property one can find a simple way to create any single quantum gate. Obviously this proposal needs the previous introduction of a new interaction that does not commute with the $\hat{x}$-axis rotation. In the rest of the section we shall find an easy procedure to approximate any single quantum gate by introducing very few modifications in the Hamiltonian.

As is presented in [10], an arbitrary 2×2 unitary transformation can be decomposed exactly as the product of rotations around two non-parallel axis plus a global phase factor. Let $\hat{n}$ and $\hat{m}$ be these axes. Then $\mathbf{U} = R_{\hat{n}}(\alpha)R_{\hat{m}}(\beta)R_{\hat{n}}(\gamma)$.[8] We have seen that for θ being an irrational multiple of 2π these rotations can be approximated applying n_i times each rotation. From this we can write, for an arbitrary quantum gate:

$$\mathbf{U} = R_{\hat{n}}(\alpha)R_{\hat{m}}(\beta)R_{\hat{n}}(\gamma) \simeq R_{\hat{n}}^{n_1}(\theta)R_{\hat{m}}^{n_2}(\theta)R_{\hat{n}}^{n_3}(\theta). \tag{41}$$

Moreover, the quantum gate can be expressed as a function of rotations around a unique axis. The corresponding needed G-transformation must take the $\hat{m}$-axis to the $\hat{n}$-axis. Finally we are led to the following expression:

$$\mathbf{U} \simeq R_{\hat{n}}^{n_1}(\theta)G^{\dagger}R_{\hat{n}}^{n_2}(\theta)GR_{\hat{n}}^{n_3}(\theta). \tag{42}$$

With this procedure we obtain rotations around $\hat{n}$. Now let us choose $\hat{n} = \hat{x}$ and $\hat{m} = \hat{y}$. The transformation $G = R_z\left(-\frac{\pi}{2}\right)$ carries $\hat{y}$ into $\hat{x}$. Taking into account the fact that our system gives us any desired rotation around $\hat{x}$, the introduction of a new interaction $R_z\left(-\frac{\pi}{2}\right)$ allows us to approximate any quantum gate. It is again the time evolution of the system itself which makes it very easy to get any single gate. What we have to do is to fix a proper value of the interaction strength a; then to apply the interaction for a time of the order t_{n_3}. After this we switch the interaction off and the transformation Z-gate acts. Next we switch on the interaction for a time t_{n_2} after which the Z-gate transformation acts again. Finally the interaction is applied for a time t_{n_1}. It could even be possible to switch on the interaction for a time $t_{n_3} + t_{n_2} + t_{n_1}$ and apply a fast transformation Z-gate in the times t_{n_3} and t_{n_2}. In this way one could program any gate with three parameters: total time and the times for the Z-gate, and reducing the external manipulation as much as possible. This leads to an approximation scheme for any single quantum gate with very few modifications on the Hamiltonian parameters.

[8] Where we have omitted the global phase factor.

The scenario that emerges after this discussion obviously leads towards the precise design of a new tool for the construction of quantum gates: the time evolution of the physical system. Using it wisely, the quantum gates can be put to work without manipulating the interaction parameters, thereby protecting the quantum computation from decoherence. Moreover, a procedure for approximating any single quantum gate is proposed with the addition of a very small number of rotational pulses. This approach would reduce the number of possible errors in the manipulation, shedding also some light on possible new ways of designing programmable quantum computers.

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