Travelling wave solutions of the generalized Benjamin-Bona-Mahony (BBM) equation by the factorization technique


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The equation describing the propagation of waves on the surface of a shallow channel was derived by J. D. Korteweg and G. de Vries in 1895.

\[ u_t - 6uu_x + u_{xxx} = 0 \]  

(KdV equation)

\[ u(x, t) = \frac{c}{2} \sec h^2 \left[ \frac{\sqrt{c}}{2} (x + ct) \right] \]
Let us consider the nonlinear second order ODE

\[
\frac{d^2 W}{d\theta^2} - \beta \frac{dW}{d\theta} + F(W) = 0
\]

\(F(W)\) is an arbitrary function of \(W\) and it can be factorized

\[
\left[ \frac{d}{d\theta} - f_2(W, \theta) \right] \left[ \frac{d}{d\theta} - f_1(W, \theta) \right] W(\theta) = 0
\]

\[
\frac{d^2 W}{d\theta^2} - \left( f_1 + f_2 + \frac{\partial f_1}{\partial W} W \right) \frac{dW}{d\theta} + f_1 f_2 W - W \frac{\partial f_1}{\partial \theta} = 0
\]
If we find a solution for this factorization problem, it will allow us to write a compatible first order ODE

\[
\begin{align*}
f_1 f_2 &= \frac{F}{W} + \frac{\partial f_1}{\partial \theta} \\
f_2 + \frac{\partial (W f_1)}{\partial W} &= \beta
\end{align*}
\]

that provides a solution to the nonlinear equation.
The generalized Benjamin-Bona-Mahony (BBM) equation

\[ u_t + u_x + a u^n u_x + u_{xxt} = 0, \quad n \geq 1 \]

- nonlinearity
- dispersion

n=1 corresponds to the BBM equation

\[ u_t + u_x + a u u_x + u_{xxt} = 0 \]

n=2 corresponds to the modified BBM equation

\[ u_t + u_x + a u^2 u_x + u_{xxt} = 0 \]
Let us assume that the generalized BBM equation has an exact solution in the form of a travelling-wave

\[ u(x, t) = \phi(\xi), \quad \xi = h \, x - \omega \, t \]

where \( h \) and \( \omega \) are real constants to be determined.

\[ \phi_{\xi\xi} - \frac{h - \omega}{h^2 \omega} \phi - \frac{a}{(n + 1) \, h \, \omega} \phi^{n+1} = -R \]

Then, we introduce the following linear transformation of the dependent and independent variables

\[ \xi = h \, \theta, \quad \phi(\xi) = \left( \frac{c \,(n + 1)}{a} \right)^{1/n} W(\theta) \]

with \( \theta = x - c \, t, \quad c = \omega / h \).
We get the nonlinear second order ODE

$$\frac{d^2 W}{d\theta^2} - W^{n+1} - k W = D,$$

where

$$k = \frac{1 - \frac{d^2 W}{d\theta^2}}{\frac{dW}{d\theta}} - \beta \left( \frac{dW}{d\theta} + F(W) = 0 \right) \right) \right)$$

$$\beta = 0 \quad \text{and} \quad F(W) = -W^{n+1} - k W - D$$

Then the concitency conditions take the form

$$f_1 f_2 = -W^n - k - D W^{-1}$$

$$f_2 = -f_1 - W \frac{\partial f_1}{\partial W}$$
The solutions of this equation are

\[ f_1 = \pm \sqrt{\frac{2 W^n}{n + 2}} + k + \frac{2 D}{W} + \frac{C}{W^2} \]

Then, replacing these solutions in the first order ODE, we get

\[ \frac{dW}{d\theta} + \sqrt{\frac{2 W^{n+2}}{n + 2}} + k W^2 + 2 DW + C = 0 \]
Another way to get this result is:

$$\frac{d^2 W}{d\theta^2} - W^{n+1} - kW = D.$$ 

can be integrated

$$(\frac{dW}{d\theta})^2 - \frac{2W^{n+2}}{n + 2} - kW^2 - 2DW = C_0$$

and then can be written as a product of functions

$$\left(\frac{dW}{d\theta} - \sqrt{\frac{2W^{n+2}}{n + 2} + k W^2 + 2DW + C_0}\right)$$

$$\left(\frac{dW}{d\theta} + \sqrt{\frac{2W^{n+2}}{n + 2} + k W^2 + 2DW + C_0}\right) = 0$$
First order ODE

\[ \frac{dW}{d\theta} + \sqrt{\frac{2 W^{n+2}}{n+2}} + k W^2 + 2 D W + C = 0 \]

The powers of \( W \) have to be integer numbers between 0 and 4, therefore \( n \in \{-1, 0, 1, 2\} \) for the integrability of first order ODE.

Let us make the transformation \( W = \varphi^p, p \neq 0, 1 \)

\[ \left( \frac{d\varphi}{d\theta} \right)^2 = \frac{2}{(n + 2) p^2} \varphi^{2+n p} + \frac{k}{p^2} \varphi^2 + \frac{2 D}{p^2} \varphi^{2-p} + \frac{C}{p^2} \varphi^{2-2p} \]
a) If \( C = D = 0 \), then \( p \in \{-2/n, -1/n, 1/n, 2/n\} \)

\[ p = 1/n \]

\[
\left( \frac{d\varphi}{d\theta} \right)^2 = \frac{2n^2}{(n + 2)} \varphi^3 + kn^2 \varphi^2
\]

\[ p = -1/n \]

\[
\left( \frac{d\varphi}{d\theta} \right)^2 = \frac{2n^2}{(n + 2)} \varphi + kn^2 \varphi^2
\]
b) If $C \neq 0$, $D = 0$, then $n = 4$

\[ p = \frac{1}{2} \]

\[
\left( \frac{d\varphi}{d\theta} \right)^2 = \frac{4}{3} \varphi^4 + 4k \varphi^2 + 4C \varphi
\]

p = -1/2

\[
\left( \frac{d\varphi}{d\theta} \right)^2 = \frac{4}{3} + 4k \varphi^2 + 4C \varphi^3
\]

c) If $C = 0$, $D \neq 0$, then no new solutions appear.
SOLUTIONS OF THE GENERALIZED BBM EQUATION

The particular solutions of the generalized BBM equation

\[ u(x, t) = \left( \frac{c(n + 1)}{a} \right)^{1/n} \varphi^p(x - c t) \]

Let us consider a quartic polynomial

\[ f(\varphi) = a_0 \varphi^4 + 4a_1 \varphi^3 + 6a_2 \varphi^2 + 4a_3 \varphi + a_4 \]

and the differential equation

\[ \left( \frac{d\varphi}{dt} \right)^2 = f(\varphi) \]
In general case the solution of the equation can be expressed in terms of the Weiestrass function $\wp(z; g_2, g_3)$

$$\varphi = \varphi_0 + \frac{1}{4} f'(\varphi_0) \left( \varphi(z; g_2, g_3) - \frac{1}{24} f''(\varphi_0) \right)^{-1}$$

with

$$z = \int_{\varphi_0}^{\varphi} \left[ f(t) \right]^{-1/2} dt$$

$$g_2 = a_0 a_4 - 4 a_1 a_3 + 3 a_2^2,$$

$$g_3 = a_0 a_2 a_4 + 2 a_1 a_2 a_3 - a_2^3 - a_0 a_3^2 - a_1^2 a_4.$$
a) If $C = D = 0$, then $p \in \{-2/n, -1/n, 1/n, 2/n\}$

$p = 1/n \quad \left(\frac{d\varphi}{d\theta}\right)^2 = \frac{2 n^2}{(n + 2)} \varphi^3 + k n^2 \varphi^2.$

The roots are: $\varphi_0 = 0$ (twice), $\varphi_0 = -k (n + 2)/2$

and nonzero solutions

$$\varphi = \frac{k(n + 2)}{4} \left(\frac{k n^2 - 12 \varphi(\theta; g_2, g_3)}{k n^2 + 6 \varphi(\theta; g_2, g_3)}\right)$$

$$g_2 = \frac{k^2 n^4}{12}, \quad g_3 = -\frac{k^3 n^6}{216}, \quad \Delta = g_2^3 - 27 g_3^2$$
we have the solution for \( 0 < c < 1 \)

\[
\varphi = -\frac{k(n + 2)}{2} \sech^2 \left[ \frac{n}{2} \sqrt{k \theta} \right]
\]

\[
= -k(n + 2) \frac{1}{1 + \cosh[n \sqrt{k \theta}]}.
\]

and another one for \( c > 1 \)

\[
\varphi = \frac{k(n + 2)}{2} \sec^2 \left[ \frac{n}{2} \sqrt{-k \theta} \right]
\]

\[
= k(n + 2) \frac{1}{1 + \cos[n \sqrt{-k \theta}]}.
\]
Solitary wave solution \( c < 1 \)

\[
u(x, t) = \left( \frac{(n + 1)(n + 2)(c - 1)}{2a} \right)^{1/n} \left( \text{sech}^2 \left[ \frac{n}{2} \sqrt{1 - \frac{c}{c}} (x - c t) \right] \right)^{1/n}
\]

Periodic solution \( c > 1 \)

\[
u(x, t) = \left( \frac{(n + 1)(n + 2)(1 - c)}{2a} \right)^{1/n} \left( \text{sec}^2 \left[ \frac{n}{2} \sqrt{\frac{c - 1}{c}} (x - c t) \right] \right)^{1/n}
\]
b) If $C \neq 0$, $D = 0$, then $n = 4$

$p = \frac{1}{2}$

\[
\left( \frac{d\varphi}{d\theta} \right)^2 = \frac{4}{3} \varphi^4 + 4k \varphi^2 + 4C \varphi.
\]

\[
\varphi = \frac{3 \varphi_0 \varphi(\theta; g_2, g_3) + 2 \varphi_0^3 + 5k \varphi_0 + 3C}{3 \varphi(\theta; g_2, g_3) - 2\varphi_0^2 - k}
\]

\[
u(x, t) = \left( \frac{c(n + 1)}{a} \right)^{1/n} \varphi^p(x - ct)
\]
SOLUTIONS OF THE BBM and MODIFIED BBM EQUATION

In this section we will consider the solutions obtained for \( n = 1, 2 \) with the integration constant \( D \neq 0 \)

\[
\frac{d^2 W}{d\theta^2} - W^{n+1} - k W = D.
\]

Let us make a simple displacement on the

\[
W(\theta) = U(\theta) + \delta
\]

\[
\frac{d^2 U}{d\theta^2} - \left( U^{n+1} + \frac{(n + 1)!}{n!} U^n \delta + \frac{(n + 1)!}{2(n - 1)!} U^{n-1} \delta^2 + ... + \frac{(n + 1)!}{n!} U \delta^n \right) = k U
\]

where the integration constant was chosen

\[
D_n = -k \delta - \delta^{n+1}
\]
a) BBM equation \((n = 1)\)

\[
\frac{d^2 U}{d\theta^2} - U^2 + (2\delta - k)U = 0
\]

where

\[
D_1 = -k\delta - \delta^2
\]

The solutions of the second order ODE for all roots

\[
U(\theta) = -\frac{5(k - 2\delta)U_0 + 4U_0^2 + 12U_0\varphi(\theta; g_2, g_3)}{(k - 2\delta) + 2U_0 - 12\varphi(\theta; g_2, g_3)}
\]

\[
g_2 = \frac{(k - 2\delta)^2}{12}, \quad g_3 = -\frac{(k - 2\delta)^3}{216} - \frac{c_1}{36}
\]

Then the solutions of the BBM equation

\[
u(x, t) = \frac{2c}{a}(U(\theta) + \delta)
\]
Choosing $C_1 = 0$, nontrivial solutions for the nonzero root

$$U_0 = -\frac{3(k - 2\delta)}{2}.$$ 

$$U(\theta) = 6 \phi(\theta + \omega; g_2, g_3) - \frac{(k - 2\delta)}{2}$$

$$g_2 = \frac{(k - 2\delta)^2}{12}, \quad g_3 = -\frac{(k - 2\delta)^3}{216}$$

Discriminant is equal to zero: $\Delta = 0$
Dark soliton solution

\[
 u(x, t) = \frac{3(c - 1)}{2a} \tanh^2 \left[ \frac{1}{2} \sqrt{\frac{c - 1}{2c}} (x - c t) \right] \quad c > 1
\]
Periodic singular solutions

\[ u(x, t) = \frac{3(1 - c)}{2a} \tan^2 \left[ \frac{1}{2} \sqrt{\frac{1 - c}{2c}} (x - c t) \right] \quad c < 1 \]
Choosing $C_1 = 0,$ and $\delta = 0,$ then $\Delta = 0$

Solitary wave (soliton) solution

Periodic solution
b) Modified BBM equation \((n = 2)\)

\[
\frac{d^2U}{d\theta^2} - (U^3 + 3 \delta U^2 + (\delta^2 + k)U) = 0 \quad \text{with} \quad D_2 = -k \delta - \delta^3
\]

\[
U(\theta) = \frac{5 K U_0 + 3 U_0^3 + 12 \delta U_0^2 + 12 U_0 \varphi(\theta; g_2, g_3)}{12 \varphi(\theta; g_2, g_3) - k - 3 (\delta + U_0)^2}
\]

\[
g_2 = \frac{K^2}{12} + \frac{C_2}{2}, \quad g_3 = -\frac{K^3}{216} + \frac{C_2 K}{12} - \frac{C_2 \delta^2}{4}
\]

The solution of the modified BBM equation

\[
u(x, t) = \sqrt{\frac{3c}{a}} (U(\theta) + \delta)
\]
Choosing $C_2 = k^2/2; \quad \delta = 0$, then $\Delta = 0$

For $c > 1$, the kink type solution

$$u(x, t) = \sqrt{\frac{3(c - 1)}{2a}} \tanh \left[ \sqrt{\frac{c - 1}{2c}} (x - ct) \right]$$
For \( c < 1 \), the periodic kink type singular solution

\[
    u(x, t) = \sqrt{\frac{3(1 - c)}{a}} \tan \left[ \sqrt{\frac{1 - c}{2c}} (x - c t) \right]
\]
The trivial choice $C_2 = 0$ and $\delta = 0$, then $\Delta = 0$

Solitary wave (soliton) solution

Periodic solution
CONCLUSIONS

- We have obtained particular solutions as well as general solutions of the generalized BBM, modified-BBM and BBM equations in terms of elliptic functions without making any ansatz.

- The factorization technique is more systematic than others previously used for the analysis of these equations. This technique gives directly solutions of the BBM equations in terms of elliptic functions.

- In this study, we have more general solutions and recovered all the solutions reported before.

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